ON THE EFFECTIVE CONE OF $\mathbb{P}^n$ BLOWN-UP AT $n+3$ POINTS

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Abstract. We compute the facets of the effective and movable cones of divisors on the blow-up of $\mathbb{P}^n$ at $n+3$ points in general position. Given any linear system of hypersurfaces of $\mathbb{P}^n$ based at $n+3$ multiple points in general position, we prove that the secant varieties to the rational normal curve of degree n passing through the points, as well as their joins with linear subspaces spanned by some of the points, are cycles of the base locus and we compute their multiplicity. We conjecture that a linear system with $n+3$ points is linearly special only if it contains such subvarieties in the base locus and we give a new formula for the expected dimension.

1. Introduction

Let $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$ denote the linear system of hypersurfaces of degree d in $\mathbb{P}^n$ passing through a collection of s points in general position with multiplicities at least $m_1, \ldots, m_s$. A classical question is to compute the dimension of $\mathcal{L}$. A parameter count provides a lower bound: the *(affine) virtual dimension* of $\mathcal{L}$ is denoted by

$$\mathrm{vdim}(\mathcal{L}) = \binom{n+d}{n} - \sum_{i=1}^{s} \binom{n+m_i-1}{n} \tag{1.1}$$

and the *(affine) expected dimension* of $\mathcal{L}$ is $\mathrm{edim}(\mathcal{L}) = \max(\mathrm{vdim}(\mathcal{L}), 0)$. If the dimension of $\mathcal{L}$ is strictly greater that the expected dimension we say that $\mathcal{L}$ is *special*.

The *dimensionality problem*, that is the classification of all special linear systems, is still open in general, in spite of intensive investigation by many authors. In particular, the study of linear systems requires information on the *effective cone* $\mathrm{Eff}_{\mathbb{R}}(X)$ of divisors on the blow-up X of $\mathbb{P}^n$ at the given points. Also the computation of the effective cone is in general a difficult task.

Let us overview now some known results. In the planar case, the Segre-Harbourne-Gimigliano-Hirschowitz Conjecture describes all effective special linear systems. It predicts that a linear system in $\mathbb{P}^2$ is special only if it contains in its base locus (-1)-curves. In particular, it conjectures the effective cone of divisors on $\mathbb{P}^2$ blown-up at points (see [12, 23, 24, 25]). On the negative side, we mention Nagata's Conjecture that predicts the nef cone of linear systems in the blown-up plane at general points. Even for the case of dimension two in spite of many partial results, both conjectures are open in general (see [13]).

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In the case of $\mathbb{P}^3$, Laface and Ugaglia Conjecture states that a Cremona reduced special linear system in $\mathbb{P}^3$ contains in its base locus either lines or the unique quadric surface determined by nine general points (see e.g. [28, 29]). The base locus lemma for the quadric in $\mathbb{P}^3$ is difficult; it is related to Nagata's Conjecture for ten points in $\mathbb{P}^2$ (see [6]). The degeneration technique introduced by Ciliberto and Miranda (see e.g. [6, 14]) is a successful method in the study of interpolation problems in higher dimensions.

In the case of $\mathbb{P}^n$ general results are rare and few things are known. The well-known Alexander-Hirschowitz Theorem [1] classifies completely the case of double points (see [7, 16, 17, 32] for more recent and simplified proofs). In general, besides the computation of some sporadic examples and the formulation of conjectures about the speciality (see e.g. [3, 4]), very little is known.

The important feature of special linear systems interpolating double points in $\mathbb{P}^n$ is that every element is singular along a positive dimensional subvariety containing the points. As conjectured by Ciliberto and Miranda (see [12, Conjecture 6.4]), it is expected that this is the case also for higher multiplicities. So it is natural to look at the *base locus* of special linear systems and try to understand the possible connection with the speciality. We call *obstructions* the subvarieties that whenever contained with multiplicity in the base locus of a linear system force it to be special.

In [5, 21] the authors started a systematic study of special linear systems from this point of view and considered in particular the case of linear obstructions. Taking into account the contribution of all the linear cycles of the base locus, a new notion of expected dimension can be given, the *linear expected dimension* $\mathrm{ldim}(\mathcal{L})$ (see Definition 2.2). We say that a system $\mathcal{L}$ is *linearly special* (resp. *linearly non-special*) if its dimension differs from (resp. equals) the linear expected dimension. In other words a linear system is linearly special if its speciality cannot be explained completely by linear obstructions. Any linear system with $s \leq n+2$ base points is linearly non-special. This was proved in [5, 21] by means of a complete cohomological classification of strict transforms in subsequently blown-up spaces of linear systems, see Section 2 for an account.

The first instance of a non-linear cycle of the base locus of a special linear system is the rational normal curve of degree n through $n+3$ general points of $\mathbb{P}^n$. The well-known Veronese Theorem (often referred to as the Castelnuovo Theorem) tells us that there exists exactly one such a curve. In $\mathbb{P}^2$ an instance of this is the unique conic through five points. In this article we focus on special linear system obstructed by rational normal curves and related varieties. We prove first a *base locus lemma* (Lemma 4.1) for linear systems with arbitrary number of general points which describes the following non-linear cycles of the base locus: the rational normal curve, its secant varieties and the cones over them. For instance, the fixed cubic hypersurface of $\mathbb{P}^4$ interpolating 7 double points, that appears as one of the exceptions in the Alexander-Hirschowitz theorem, is the variety of secant lines to the rational normal curves given by the seven points.

We expect that when the multiplicity of containment in the base locus is high enough with respect to the degree, those cycles forces the linear system to be special. More precisely, we give a conjectural formula (Definition 6.1, Conjecture 6.4) for the dimension of linear systems based at $n+3$ points. The formula in Definition 6.1 takes into account the contribution of the linear cycles and also that of the normal curves and related cycles in the base locus. In Section 6.2 we prove

that this conjecture holds for $n = 2, 3$ and for general n in a number of interesting families of homogeneous linear systems.

As a consequence of our analysis, we deduce the main result of this paper, that is an explicit description of all effective divisors in X, the blown-up $\mathbb{P}^n$ at $n+3$ points. We give in Theorem 5.1 a list of inequalities that define the effective cone of X, and we also describe the movable cone of X, Theorem 5.3.

We mention that a new approach to the dimensionality problem for $s = n+3$ points was introduced in [35] and their analysis relies on sagbi bases. For $s \leq n+3$, in [10, 31] it was proved that the blow-up of $\mathbb{P}^n$ at s points in general position is a *Mori dream space*. In particular Castravet and Tevelev [10] gave the rays of the effective cone, see Section 5.1 for more details. What is interesting is the fact that Castravet and Tevelev's extremal rays can be formulated in terms of hypersurfaces that are either secant varieties to the rational normal curve through the $n+3$ points or their joins with linear subspaces spanned by the points, see Section 3. In this paper we show that in fact both the effective cone (Theorem 5.1) and movable cone (Theorem 5.3) of the blown-up space, and the dimensionality problem, depend exclusively on these secant varieties seen as cycles of arbitrary codimension in $\mathbb{P}^n$. Moreover there is a bijection between the 2^{n+2} weights of the half-spin representations of $\mathfrak{so}_{2(n+3)}$ and the generators of the Cox ring of the blow-up of $\mathbb{P}^n$ at $n+3$ points in general position (see [20, 10, 34]). In [34] in particular this bijection interprets the latter space as a spinor variety. It would be interesting to establish a dictionary that translates the language of secant varieties into that of spinor varieties.

The article is organized as follows. In Section 2 we give an account on the notion of linear speciality [5, 21].

In Section 3 we give a geometric description of the rational normal curves and (cones over) their secants and we give an interpretation in terms of divisors of those among them that are of codimension 1, by means of Cremona transformations of $\mathbb{P}^n$. In particular, these divisors are the Castravet-Tevelev rays generating the effective cone.

In Section 4 we prove the base locus lemma for rational normal curves and related cycles, Lemma 4.1.

In Section 5 we describe the effective and movable cones of X, Theorem 5.1 and Theorem 5.1.

In Section 6 we introduce the new notion of expected dimension, σldim (Definition 6.1), and state our Conjecture 6.4, exhibiting a list of evidences in Section 6.2.

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2. Linear speciality of linear systems

Given a linear system $\mathcal{L}$, we say that *its base locus contains a subvariety* L *with multiplicity* k if any hypersurface in $\mathcal{L}$ has multiplicity at least k along each point of L.

Let $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$ be a non-empty linear system, let $I(r) \subseteq \{1, \ldots, s\}$ be any multi-index of length $|I(r)| = r + 1$, for $0 \leq r \leq \min(n, s) - 1$ and denote by $L_{I(r)}$ the unique r-linear cycle through the points p_i, for $i \in I(r)$. Set

$$k_{I(r)} = \max\left(\sum_{i \in I(r)} m_i - rd, 0\right). \tag{2.1}$$

It is an easy consequence of Bézout's Theorem that if $k_{I(r)} > 0$ then all elements of $\mathcal{L}$ vanish along $L_{I(r)}$.

In [5] a (sharp) base locus lemma for linear cycles, that we will refer to as *linear base locus lemma*, for linear systems with at most $n + 2$ points was proved and later generalized in [21] to linear systems with arbitrary numbers of points. We summarize the content of the two above mentioned results in the following

Lemma 2.1 ([21, Proposition 4.2])**.** *For any non-empty linear system $\mathcal{L}$ with arbitrary number of points and for any $0 \leq r \leq n - 1$, the multiplicity of containment of the cycle $L_{I(r)}$ in the base locus of $\mathcal{L}$ is $k_{I(r)}$.*

When the order of vanishing is high, precisely when $k_{I(r)} > r$, then $L_{I(r)}$ provides obstruction to the non-speciality. This observation yields the following definition of expected dimension.

Definition 2.2 ([5, Definition 3.2])**.** The *(affine) linear virtual dimension* of $\mathcal{L}$ is the number

$$\sum_{r=-1}^{s-1} \sum_{I(r) \subseteq \{1, \ldots, s\}} (-1)^{r+1} \binom{n + k_{I(r)} - r - 1}{n}. \tag{2.2}$$

where we set $I(-1) = \emptyset$ and $k_{I(-1)} = d$.

The *(affine) linear expected dimension* of $\mathcal{L}$, denoted by $\mathrm{ldim}(\mathcal{L})$, is 0 if $\mathcal{L}$ is contained in a linear system whose linear virtual dimension is negative, otherwise is the maximum between the linear virtual dimension of $\mathcal{L}$ and 0.

In (2.2), the number $(-1)^{r+1}\binom{n+k_{I(r)}-r-1}{n}$ computes the contribution of the linear cycle $L_{I(r)} \cong \mathbb{P}^r$ spanned by the points p_{i_j}, $i_j \in I(r)$. If all the numbers $k_{I(r)}$ are zero, the linear virtual dimension (2.2) equals the virtual dimension (1.1) of $\mathcal{L}$. Asking whether the dimension of a given linear system equals its linear expected dimension is a refinement of the classical question of asking whether the dimension equals the expected dimension. We say that a linear system $\mathcal{L}$ is *linearly special* if $\dim(\mathcal{L}) \neq \mathrm{ldim}(\mathcal{L})$. On the other hand a linear system is called *linearly non-special* (or *only linearly obstructed*) if its dimension equals the linear expected dimension.

We recall here, for the reader convenience, the following results on linearly speciality and effectiveness.

Theorem 2.3 ([5, Corollary 4.8, Theorem 5.3])**.** *All non-empty linear systems with $s \leq n + 2$ points are linearly non-special.*

Moreover, for $s \geq n+3$ let $s(d) \geq 0$ is the number of points of multiplicity d. If

$$\sum_{i=1}^{s} m_i \leq nd + \min(n - s(d), s - n - 2), \quad 1 \leq m_i \leq d, \tag{2.3}$$

then $\mathcal{L}$ is linearly non-special.

Theorem 2.4 ([5, 8, 10])**.** *If $s \leq n+2$, then $\mathcal{L}$ is non-empty if and only if*

$$m_i \leq d, \ \forall i = 1 \ldots, s, \quad \sum_{i=1}^{s} m_i \leq nd. \tag{2.4}$$

Moreover if $s \geq n+3$ and (2.3) is satisfied, then $\mathcal{L}$ is non-empty.

Since Theorem 2.4 gives a statement about linear systems we assumed that the coefficients d and m_i are positive. In order to translate the statement into the language of divisors on the blow-up of $\mathbb{P}^n$ at points we relax the positivity assumption on the coefficients m_i's and we obtain the following

Corollary 2.5. *The effective cone of $\mathbb{P}^n$ blown-up at $s \leq n+2$ points in general position is described by (2.4) and the inequalities*

$$d \geq 0, \quad \sum_{i \in I} m_i \leq nd, \ \forall I \subseteq \{1, \ldots, s\}, \ |I| = n+1. \tag{2.5}$$

We remark that (2.3) is also sufficient condition for the base locus of $\mathcal{L}$ to not contain any multiple rational normal curve. In Section 4 we will give a sharp base locus lemma for the rational normal curve for all linear systems based at $n+3$ general points. Moreover in Section 5 we will give necessary and sufficient conditions for a linear system in $\mathbb{P}^n$ based at $n+3$ general points to be non-empty.

2.1. **Connection to the Fröberg-Iarrobino Conjecture.** The problem of determining the dimension of linear systems with assigned multiple points is related to the Fröberg-Iarrobino Weak and Strong Conjectures [22, 27], which give a predicted value for the Hilbert series of an ideal generated by s general powers of linear forms in the polynomial ring with $n+1$ variables. Such an ideal corresponds, via apolarity, to the ideal of a collection of fat points, therefore it is possible to give a geometric interpretation of this conjecture, as Chandler pointed out [18]. See also [5, Sect. 6.1] for more details.

In terms of our Definition 2.2 the Weak Conjecture can be stated as follows: the dimension of a *homogeneous* linear system, i.e. one for which all points have the same multiplicity, is bounded below by its linear expected dimension.

Conjecture 2.6 (Weak Fröberg-Iarrobino Conjecture)**.** *The linear system $\mathcal{L} = \mathcal{L}_{n,d}(m^s)$ satisfies $\dim(\mathcal{L}) \geq \operatorname{ldim}(\mathcal{L})$.*

Moreover, the Strong Conjecture states that a homogeneous linear system is always linearly non-special besides a list of exceptions.

Conjecture 2.7 (Strong Fröberg-Iarrobino Conjecture)**.** *The linear system $\mathcal{L} = \mathcal{L}_{n,d}(m^s)$ satisfies $\dim(\mathcal{L}) = \operatorname{ldim}(\mathcal{L})$ except perhaps when one of the following conditions holds:*

(i) $s = n+3$;
(ii) $s = n+4$;
(iii) $n = 2$ *and* $s = 7$ *or* $s = 8$;

(iv) $n = 3$, $s = 9$ *and* $d \geq 2m$;
(v) $n = 4$, $s = 14$ *and* $d = 2m$, $m = 2$ *or* 3.

Notice that in [18] the formulation of case *(iv)* requires the condition $d = 2m$, anyway it is known (see for example [6]) that for any degree $d \geq 2m$ the linear system $\mathcal{L}_{n,d}(m^s)$ is linearly special for m high enough.

In this paper we conjecture that linear systems with $n+3$ points are linearly special only if they contain in their base locus the rational normal curve given by the $n+3$ points, its secant varieties or their joins with linear subspaces spanned by subsets of the set of the $n+3$ points (Conjecture 6.4). Moreover, we give a new definition of expected dimension, the *secant linear dimension* σldim (see Definition 6.1), that provides a correction term for ldim. In particular, in the homogeneous case this completes the Strong Fröberg-Iarrobino Conjecture.

Remark 2.8. It would be interesting to extract the Hilbert series of ideals generated by $n+3$ powers of linear forms from our formula of σldim, Definition 6.1.

2.2. **General vision.** We can interpret the base locus lemma for rational normal curves and related cycles −that we prove in Section 4− and the definition of σldim both as extensions of the results contained in our previous works [5, 21]: Lemma 2.1 and Definition 2.2.

We expect rational normal curves and related cycles to appear as obstruction to the non-speciality also in the case of linear systems with $s \geq n+3$ base points. A natural generalization of Conjecture 2.6 would be that σldim provides a lower bound for the dimension of any general non-homogeneous linear system. We plan to further investigate this problem. We chose to dedicate this work to the case of $n+3$ points because it is the first case where non-linear obstructions appear and was not understood before for the general case of $\mathbb{P}^n$.

3. Secant varieties to rational normal curves and Cremona transformations

In this section we collect a series of well-known geometric aspects of secant varieties to rational normal curves. The first important point is the following.

Theorem 3.1 (Veronese)**.** *There exists a unique rational normal curve of degree n passing through $n+3$ points in general position in $\mathbb{P}^n$.*

This theorem is classically known and its first proof is due to Veronese [37], although it is often attributed to Castelnuovo.

In this section and throughout this paper we will adopt the following notation. Let $p_1, \ldots, p_{n+3} \in \mathbb{P}^n$ general points, let C be the rational normal curve of degree n interpolating them and, for every $t \geq 1$, let $\sigma_t := \sigma_t(C) \subset \mathbb{P}^n$ be the variety of t-secant $\mathbb{P}^{t-1}$'s to C. In this notation we have $\sigma_1 = C$.

Rational normal curves are never secant defective; in particular we have the following formula for the secant dimension:

$$\dim(\sigma_t) = \min(n, 2t-1).$$

Moreover rational normal curves are of minimal secant degree if $2t-1 < n$, see [15]:

$$\deg(\sigma_t) = \binom{n-t+1}{t}.$$

Secant varieties are highly singular, in particular for $t \geq 2$, $2t-1 < n$, we have $\sigma_{t-1} \subset \mathrm{Sing}(\sigma_t)$. Moreover the multiplicity of σ_t along σ_τ, for all $1 \leq \tau < t$, satisfies the following (see e.g. [15])):

$$\mathrm{mult}_C(\sigma_t) = \binom{n-t}{t-1}, \qquad \mathrm{mult}_{\sigma_\tau}(\sigma_t) \geq \binom{n-t-\tau+1}{t-\tau}.$$

3.1. **Cones over the secant varieties to the rational normal curve.** In this section, we consider cones over the σ_t with vertex spanned by a subset of the base points. Let $I \subset \{1, \ldots, n+3\}$ with $|I| = r+1$. We use the conventions $|\emptyset| = 0$ and $\sigma_0 = \emptyset$. Let us denote by

$$\mathrm{J}(L_I, \sigma_t) \tag{3.1}$$

the join of L_I and σ_t.

Recall that $\sigma_t = \mathrm{J}(\sigma_{t-1}, C) = \mathrm{J}(\sigma_{t-2}, \sigma_2)$ etc. Notice also that $\mathrm{J}(L_I, \sigma_t) \subset \sigma_{|I|+t}$.

The dimensions of such joins can be easily computed:

$$r_{I,\sigma_t} := \dim(\mathrm{J}(L_I, \sigma_t)) = \dim(L_I) + \dim(\sigma_t) + 1 = |I| + 2t - 1. \tag{3.2}$$

3.2. **Divisorial cones.** When $\mathrm{J}(L_I, \sigma_t)$ is a hypersurface, namely when $r_{I,\sigma_t} = n-1$ that is I is such that $|I| = n-2t$, we can characterize these cones as the unique section of a certain linear system of hypersurfaces of $\mathbb{P}^n$ interpolating points $p_1, \ldots, p_{n+3}$ with multiplicity.

We will denote by $\mathcal{L}_{n,d}(m_1, \ldots, m_s)$ the linear system of degree$-d$ hypersurfaces of $\mathbb{P}^n$ interpolating the $n+3$ points with multiplicity $m_1, \ldots, m_s$ respectively.

We first discuss the case when σ_t is a hypersurface. Precisely, when $n = 2t$, $I = \emptyset$, we have that σ_t is a degree $(t+1)$ hypersurface with multiplicity t along C and in particular at the fixed points $p_1, \ldots, p_{n+3}$. In this notation we have that σ_t belongs to the the linear system $\mathcal{L}_{2t,t+1}(t^{2t+3})$. Moreover one can prove that it is the only element satisfying the interpolation condition, see also Section 6.2.3 (Proposition 6.12). For instance for $t = 1$ one obtains the plane conic through five points, $\mathcal{L}_{2,2}(1^5)$, for $t = 2$ one obtains $\mathcal{L}_{4,3}(2^7)$.

Remark 3.2. In Section 4 (Corollary 4.3), we will show that, when $n = 2t$, σ_t has multiplicity exactly $t - \tau + 1$ on σ_τ, for all $1 \leq \tau < t$.

Assume now that σ_t has higher codimension in $\mathbb{P}^n$. Fix I such that $|I| = n-2t \geq 1$ and consider $\pi_I : \mathbb{P}^n \dashrightarrow \mathbb{P}^{2t}$ the projection from the linear subspace L_I. Denote by $C' := \pi_I(C)$ the projection of C and $\sigma'_t := \pi_I(\sigma_t(C))$ the projection of its t-secant variety. Then C' is a rational normal curve of degree $2t$ and $\sigma'_t = \sigma_t(C')$ is the t-secant variety to C'. Hence the hypersurface $\mathrm{J}(L_I, \sigma_t)$ is the cone with vertex the linear subspace L_I over the secant variety σ'_t.

We conclude that for any I such that $|I| \geq 0$, the following formula holds:

$$\mathrm{J}(L_I, \sigma_t) = \mathcal{L}_{n,t+1}((t+1)^{n-2t}, t^{2t+3}). \tag{3.3}$$

We mention that in [10, Theorem 2.7] the authors prove that divisors of the form (3.3) are the rays of the effective cone $\mathrm{Eff}_{\mathbb{R}}(X)$ (see also Section 5.1).

3.3. **The standard Cremona transformation.** We recall that the *standard Cremona transformation* of $\mathbb{P}^n$ is the birational transformation defined by the following rational map:

$$\mathrm{Cr} : (x_0 : \cdots : x_n) \to (x_0^{-1} : \cdots : x_n^{-1}),$$

see e.g. [20]. Let $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \dots, m_s)$ be a linear system based on s points in general position; we can assume, without loss of generality, that the first $n+1$ are the coordinate points. The map Cr can be seen as the morphism associated to the linear system $\mathcal{L}_{n,n}((n-1)^{n+1})$. This induces an automorphism of the Picard group of the n-dimensional space blown-up at s points by sending the strict transform of $\mathcal{L}_{n,d}(m_1, \dots, m_s)$ into the strict transform of

$$\mathrm{Cr}(\mathcal{L}) := \mathcal{L}_{n,d-c}(m_1 - c, \dots, m_{n+1} - c, m_{n+2}, \dots, m_s)$$

where

$$c := m_1 + \cdots + m_{n+1} - (n-1)d.$$

We have the following equality

$$\dim(\mathcal{L}) = \dim(\mathrm{Cr}(\mathcal{L})). \tag{3.4}$$

If $c \leq 0$ we will say that the linear system $\mathcal{L}$ is *Cremona reduced*.

Remark 3.3. One can check that the join divisor $\mathrm{J}(L_I, \sigma_t)$ is in the orbit of the Weyl group of an exceptional divisor. To see this, order the multiplicities decreasingly and apply the Cremona action to the first $n+1$ points $t+1$ times. Indeed,

$$c(\mathcal{L}_{n,t+1}((t+1)^{n-2t}, t^{2t+3})) = 1$$

therefore

$$\mathrm{Cr}(\mathcal{L}_{n,t+1}((t+1)^{n-2t}, t^{2t+3})) = \mathcal{L}_{n,t}(t^{n-2t+2}, (t-1)^{2t+1}).$$

This proves the claim since one can recursively replace t by $t-1$ until $t = 0$.

4. Base locus lemma

In this section we give a sharp base locus lemma for the rational normal curve, and (cones over) its secant varieties, that generalizes Lemma 2.1 from the case of at most $n+2$ points to the case of arbitrary number of points s.

If $s \geq n+3$, as in Section 3, we denote by C the unique rational normal curve through any subset of $n+3$ points, say $p_1, \dots, p_{n+3}$, and by σ_t its t-th secant variety. We denote by $J(I, \sigma_t)$ the join between any index set $I \subset \{1, \dots, s\}$ and σ_t, and by r_{I,σ_t} its dimension. For $s = n+3$, this notions coincide with the ones introduced in (3.1) and (3.2).

To a linear system $\mathcal{L}_{n,d}(m_1, \dots, m_{n+3}, \dots, m_s)$ we associate the following integers:

$$k_C := \sum_{i=1}^{n+3} m_i - nd, \tag{4.1}$$

$$k_{I,\sigma_t} := \sum_{i \in I} m_i + tk_C - (|I| + t - 1)d. \tag{4.2}$$

Notice that, by setting

$$M := \sum_{i=1}^{n+3} m_i,$$

one can write

$$k_{I,\sigma_t} = tM + \sum_{i \in I} m_i - ((n+1)t + |I| - 1)d. \tag{4.3}$$

Moreover, if in (4.2) we replace $t=0$ we obtain

$$k_I := \sum_{i\in I} m_i - (|I|-1)d$$

(cfr. (2.1)); if $|I|=0$ and $t=1$ we obtain $k_C := k_{\emptyset,\sigma_1} = M - nd$; if $|I|=0$ we obtain $k_{\sigma_t} := tk_C - (t-1)d$.

The number k_{σ_t} is the multiplicity of containment of a t-secant $\mathbb{P}^{t-1}$ to C in the base locus of $\mathcal{L}$. This is a straightforward consequence of the linear base locus lemma, knowing that k_C is the multiplicity of containment of C. In the next lemma we prove that in fact the whole σ_t is contained in the base locus with that multiplicity.

Lemma 4.1 (Base locus lemma)**.** *Let $\mathcal{L}$ be an effective linear system with s base points. In the same notation as above, let C be the rational normal curve given by $n+3$ of them, fix any $I\subset\{1,\dots,s\}$ and $t\geq 0$ such that $r_{I,\sigma_t}\leq n-1$.*

If $k_{I,\sigma_t}\geq 1$, then the cone $\mathrm{J}(L_I,\sigma_t)$ is contained in the base locus with exact multiplicity k_{I,σ_t}.

Proof. Since all of the results used in this proof hold for arbitrary number of points s, it is enough to prove that statement for $s=n+3$ and for the corresponding C.

If $t=0$ then $\mathrm{J}(L_I,\sigma_t)=L_I$ and the statement follows from Lemma 2.1.

Assume that $I=\emptyset$ and $t=1$. Then $\mathrm{J}(L_I,\sigma_t)=C$ is the rational normal curve through the $n+3$ points. In [9, Theorem 4.1], the authors prove that performing the Cremona transformation based at the first $n+1$ base points of $\mathcal{L}$, then C is mapped to the line through the last two points p_{n+2} and p_{n+3}, that we may denote by $L_{I(1)}$. Let K_C be the multiplicity of containment of C in $\mathcal{L}$: one has $K_C\geq k_C$ by Bézout's Theorem. Observe that K_C is also the multiplicity of containment of the line $L_{I(1)}$ in $\mathrm{Cr}(\mathcal{L})$. We conclude by noticing that by the linear base locus lemma, this is given by

$$K_C = k_{I(1)} = m_{n+2} + m_{n+3} - \left(nd - \sum_{i=1}^{n+1} m_i\right) = M - nd = k_C.$$

Assume that $I=\emptyset$ and $t\geq 2$, $2t-1<n$. The above parts imply that any secant $(t-1)$-plane spanned by t distinct points of C is contained in the base locus of $\mathcal{L}$ with multiplicity exactly k_{σ_t}. Moreover, since the multiplicity is semi-continuous, it follows that all limits of t-secant $(t-1)$-planes are contained in the base locus with multiplicity at least k_{σ_t}. Hence the secant variety σ_t has multiplicity k_{σ_t}.

Finally, the case $I\neq\emptyset$, $t\geq 1$ follows from the above. Indeed every line L in $\mathrm{J}(L_I,\sigma_t)$ connecting a point of the vertex L_I and a point of the base σ_t, is contained in the base locus of $\mathcal{L}$ with multiplicity $k_I + k_{\sigma_t} - d$. □

Remark 4.2. Notice that the effectivity of $\mathcal{L}$ implies the following inequality $k_C\leq m_i\leq d$, for all i. Indeed if $k_C>m_i$ for some i, then $\sum_{j\neq i} m_j > nd$, a contradiction by Theorem 5.1. This in particular implies $k_{\sigma_{|I|+t}}\leq k_{I,\sigma_t}\leq k_I$. Moreover the obvious equality $k_{I,\sigma_t} = \sum_{i\in I} m_i - |I|d + k_{\sigma_t}$ and the effectivity condition $m_i\leq d$ imply $k_{I,\sigma_t}\leq k_{\sigma_t}$.

Because of the containment relations $L_I,\sigma_t\subseteq \mathrm{J}(L_I,\sigma_t)\subseteq\sigma_{|I|+t}$, the above inequalities read as: if L_I or σ_t is not in the base locus of $\mathcal{L}$, neither is $\mathrm{J}(L_I,\sigma_t)$ nor $\sigma_{|I|+t}$; if $\mathrm{J}(L_I,\sigma_t)$ is not contained in the base locus, neither is $\sigma_{|I|+t}$.

4.1. **Geometric consequences of the base locus lemma.** An immediate consequence of the base locus lemma is a description of the singularities of the secant variety, whenever this is a hypersurfaces. Indeed since σ_t is the unique element of the linear system $\mathcal{L}_{2t,t+1}(t^{2t+3})$, one can compute the multiplicity along the lower order secant varieties.

Corollary 4.3. *Let $n = 2t$ and $1 \leq \tau \leq t$. Then σ_t is singular with multiplicity $t - \tau + 1$ on $\sigma_\tau \setminus \sigma_{\tau-1}$*

Another consequence of Lemma 4.1 is the following result that in particular implies that Cremona reduced linear systems are movable.

Corollary 4.4. *Let $\mathcal{L}$ be an effective linear system with arbitrary number of points. Assume that $\mathcal{L}$ is Cremona reduced. Then $\mathcal{L}$ does not contain any divisorial component of type $\mathrm{J}(I(n-2t-1), \sigma_t)$ in its base locus.*

Proof. Write $n = 2l + \epsilon$, with $\epsilon \in \{0, 1\}$. By Lemma 4.1, it is enough to prove that $k_{I(n-2t-1),\sigma_t} \leq 0$ for all $0 \leq t \leq l + \epsilon$.

Since $\mathcal{L}$ is Cremona reduced, the hyperplane spanned by the collection of points parametrized by $I(n-1)$ is not contained in the base locus, for any $I(n-1)$. Indeed if $I(n-1) \subset I(n)$, for some $I(n)$, we have

$$k_{I(n-1)} < \sum_{i \in I(n)} m_i - (n-1)d \leq 0.$$

This proves the statement for $t = 0$.

Assume $1 \leq t \leq l + \epsilon$. For any fixed index set $I := I(n-2t-1)$ of cardinality $n-2t$, choose $2t$ distinct indices in its complement: $\{i_1, \ldots, i_{2t}\} \subset \{1, \ldots, n+3\} \setminus I$. We have

$$\begin{aligned} k_{I,\sigma_t} &= tM + \sum_{i \in I} m_i - (t+1)(n-1)d \\ &= \sum_{j=1}^{s} \big(M - m_{i_j} - m_{i_{j+t}} - (n-1)d\big) + \left(\sum_{j=1}^{2t} m_{i_j} + \sum_{i \in I} m_i - (n-1)d \right) < 0 \end{aligned}$$

The first t terms are negative by assumption, the last is strictly negative because of the hyperplane case $t = 0$. $\square$

Remark 4.5. In Section 5.1 we will see that effective divisors in the blown-up $\mathbb{P}^n$ at $n+3$ general points without fixed components of type $\mathrm{J}(I(n-2t-1), \sigma_t)$, namely those satisfying $k_{I(n-2t-1),\sigma_t} \leq 0$, are movable. Hence the cone of Cremona reduced effective divisors, that is polyhedral since defined by inequalities, is contained in the movable cone, that is in turn contained in the effective cone.

5. Effective and movable cones

In this section we will give necessary and sufficient conditions for linear systems $\mathcal{L}_{n,d}(m_1, \ldots, m_{n+3})$ in $\mathbb{P}^n$ with $n+3$ base points in general position to have at least one section. This is equivalent to an *effectivity theorem* for divisors on X, the blown-up $\mathbb{P}^n$ at $n+3$ general points, and provides a generalization of Theorem 2.4 and Corollary 2.5.

Throughout this section, we will use the same notation introduced in Section 4. Moreover we let $\mathrm{N}^1(X)$ denote the Neron-Severi group of X with coordinates

$(d, m_1, \ldots, m_{n+3})$ corresponding to the hyperplane class and the classes of the exceptional divisors.

Theorem 5.1 (Effectivity Theorem)**.** *(I) For $n \geq 2$, a linear system $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_{n+3})$ is non-empty if and only if*

$$\begin{array}{lcr} (A_n) & m_i \leq d, & \forall i = 1, \ldots, n+3, \\ (B_n) & M - m_i \leq nd, & \forall i = 1, \ldots, n+3, \\ (C_{n,t}) & k_{I,\sigma_t} \leq 0, & \forall |I| = n-2t+1,\ 1 \leq t \leq l+\epsilon, \end{array}$$

where $n = 2l + \epsilon$, $\epsilon \in \{0, 1\}$.

(II) The facets of the effective cone of divisors $\mathrm{Eff}_{\mathbb{R}}(X)$ are given by the equalities in (A_n), (B_n), and $(C_{n,t})$, with $-1 \leq t \leq l + \epsilon$.

Remark 5.2. In Section 4 the numbers k_{I,σ_t} (4.2) appeared as the multiplicities of containment in the base locus, of cycles of codimension at least one, namely for $1 \leq t \leq l + \epsilon$ and $0 \leq |I| \leq n - 2t - 1$. In this section, the numbers $k_{I(n-2t),\sigma_t}$ appearing in $(C_{n,t})$ (Theorem 5.1) are formal generalizations of the above to the zero codimensional case: $\mathrm{J}(I(n-2t), \sigma_t) = \mathbb{P}^n$. In analogy with the cases $s = n+1, n+2$ points, where the condition $\sum_{i=1}^{s} m_i \leq nd$ corresponds to asking that the linear span of all of the points $-$ that is the whole space $\mathbb{P}^n$ $-$ is not in the base locus (see [21, Theorem 1.6]), here we ask that no n-dimensional *virtual* cycle $\mathrm{J}(I(n-2t), \sigma_t)$ is in the base locus of linear systems with $n + 3$ points.

The proof of Theorem 5.1 is by induction on n. For this reason we think it is convenient to treat the initial case, $n = 2$, separately. Here the conditions of part *(I)* read as follows.

$$\begin{array}{lcr} (A_2) & m_i \leq d, & \forall i = 1, \ldots, 5, \\ (B_2) & M - m_i \leq 2d, & \forall i = 1, \ldots, 5, \\ (C_{2,1}) & M + m_i \leq 3d, & \forall i = 1, \ldots, 5, \end{array}$$

Proof of Theorem 5.1, part (I), case $n = 2$. Without loss of generality we may assume $m_1 \geq m_2 \geq \cdots \geq m_5 \geq 1$. It is enough to prove that $\mathcal{L}$ is non-empty if and only if

$$m_1 \leq d, \quad m_1 + m_2 + m_3 + m_4 \leq 2d, \quad 2m_1 + m_2 + m_3 + m_4 + m_5 \leq 3d.$$

If $\mathcal{L}$ is non-empty, then obviously $m_1 \leq d$. Moreover since $\mathcal{L}_{2,d}(m_1, \ldots, m_4)$ is non-empty, then $m_1 + m_2 + m_3 + m_4 \leq 2d$.

To prove the third inequality assume first that $k_C = \sum_{j=1} m_j - 2d \leq 0$. Then $2m_1 + m_2 + m_3 + m_4 + m_5 - 3d = k_C + m_1 - d \leq 0$. If $k_C \geq 1$, then by Lemma 4.1, the conic C is a fixed component of $\mathcal{L}$ and the residual part has degree $d' := d - 2k_C = 5d - 2\sum_{j=1}^{5} m_j$ and multiplicities $m'_i = 2d - \sum_{j=1}^{5} m_j + m_i$ at the 5 points. Notice that $m'_i \geq 0$ by the second inequality. Effectivity implies $d' \geq m'_1$ and this is equivalent to the third inequality.

We now prove the other implication. If $k_C = \sum_{j=1}^{5} m_j - 2d \leq 0$ then $\mathcal{L}$ is non-empty by Theorem 2.4. Assume $k_C \geq 1$. By Lemma 4.1 the conic through the five points is contained in the base locus with multiplicity k_C. Notice that $m_1 + m_2 + m_3 + m_4 \leq 2d$ implies $k_C \leq m_5$. The residual is $\mathcal{L}'$ with $d' = d - 2k_C =$

$5d-2\sum_{j=1}^5 m_j$ and $m'_i=m_i-k_C=2d-\sum_{j=1}^5 m_j+m_i\geq 0$, for all $i=1,\dots,5$. Obviously $\sum_{j=1}^5 m'_j-2d'=0$. We claim $d'\geq m'_i$, for all $i=1,\dots,5$. Hence $\mathcal{L}'$ is effective. To prove the claim for $i=1$, notice that $d'-m'_1=5d-2\sum_{j=1}^5 m_j-2d+\sum_{j=1}^5 m_j-m_1=3d-\sum_{j=1}^5 m_j-m_1\geq 0$. □

We now complete the proof of the first part of effectivity theorem for $n\geq 3$.

Proof of Theorem 5.1, part (I), $n\geq 3$. Without loss of generality, we may reorder the points so that $m_1\geq\cdots\geq m_{n+3}$. If $m_{n+3}=0$, the set of conditions (B_n) becomes just $\sum_{j=1}^{n+2} m_j\leq nd$ and the third set of conditions, $(C_{n,t})$ is redundant. In this case the result was proved in [5, Lemma 2.2], see Theorem 2.4. Hence we will assume $m_{n+3}\geq 1$.

"Only if" implication:

If $\mathcal{L}$ is effective then (A_n) and (B_n) trivially hold.

The expanded expressions of condition $(C_{n,t})$ is

$$k_{I,\sigma_t}=tM+\sum_{i\in I}m_i-((t+1)n-t)d\leq 0,$$

for all $1\leq t\leq\left\lfloor\frac{n-1}{2}\right\rfloor=l+\epsilon$ and all multi-index $I=I(n-2t)$, see (4.3). Fix such t and $I=I(n-2t)$. Take any $j\in I(n-2t)$ and denote by $I\setminus\{j\}$ its complement in I. Write

$$k_{I\setminus\{j\},\sigma_t}=tM+\sum_{i\in I\setminus\{j\}}m_i-((t+1)n-(t+1))d.$$

In order to prove the inequality $(C_{n,t})$, we consider the following cases.

Case (1). Assume that $k_{I\setminus\{j\},\sigma_t}\leq 0$. Since $k_{I,\sigma_t}=k_{I\setminus\{j\},\sigma_t}+(m_j-d)$, we conclude by (A_n).

Case (2). Assume that $k_{I\setminus\{j\},\sigma_t}\geq 1$. By Lemma 4.1, $k_{I\setminus\{j\},\sigma_t}$ is the multiplicity of containment of the cone $\mathrm{J}(I\setminus\{j\},\sigma_t)$, that is the hypersurface $\mathcal{L}_{n,t+1}((t+1)^{n-2t},t^{2t+3})$, see (3.3). The residual of $\mathcal{L}$ after its removal is still effective by assumption. Let us denote by d' and by m'_j the degree and the multiplicity at the point p_j of the residual, that is $d'=d-(t+1)k_{I\setminus\{j\},\sigma_t}$ and $m'_j=m_j-tk_{I\setminus\{j\},\sigma_t}$. Effectivity implies that $d'\geq m'_j\geq 0$. We conclude by noticing that $d'-m'_j\geq 0$ is equivalent to $k_{I,\sigma_t}\leq 0$.

"If" implication:

The proof is by induction on n, with initial step the case $n=2$ for which the statement is already proved to hold. Assume the statement true for $n-1$.

We will construct recursively an element that belongs to $\mathcal{L}$, hence proving nonemptiness. We will treat the following cases and subcases separately.

- (0) $k_C\leq 0$.
- (1) $k_C\geq 1$ and $m_1=d$.
- (2) $k_C\geq 1$ and $m_1\leq d-1$,
  - (2.a) $\mathcal{L}$ is Cremona reduced,
  - (2.b) $\mathcal{L}$ is not Cremona reduced.

Case (0). In this case $\mathcal{L}$ is effective by Theorem 2.4.

Case (1). Notice that the elements of $\mathcal{L}$, cones with vertex at p_1, are in bijection with the elements of a linear system $\mathcal{L}' = \mathcal{L}_{n-1,d}(m_2, \ldots, m_{n+3})$. One can check easily that $\mathcal{L}'$ satisfies conditions (A_{n-1}), (B_{n-1}), being those implied by (A_n), (B_n) respectively. Moreover for $1 \leq t \leq l$ and any index set $I = I(n-2t)$ such that $1 \in I$, condition $(C_{n,t})$ implies condition $(C_{n-1,t})$, for the index set $I' = I \setminus \{1\} = I(n-1-2t)$. Indeed we have

$$\begin{aligned}
k_{I',\sigma_t} &= tM' + \sum_{i\in I'} m_i' - ((t+1)(n-1)-t)d' \\
&= tM + \sum_{i\in I} m_i - (t+1)m_1 - ((t+1)n-t)d + (t+1)d \\
&= k_{I,\sigma_t} \leq 0.
\end{aligned}$$

Since $\mathcal{L}'$ is effective, then $\mathcal{L}$ is.

Case (2.a). Notice that in this case $m_{n+3} \geq 2$. Set $I := \{1, \ldots, n-2\}$ and consider the cone $\mathrm{J}(I, C)$ over the rational normal curve C with vertex the linear subspaces L_I spanned by the first $n-2$ points. As in (3.3), $\mathrm{J}(I, C)$ can be interpreted as the fixed divisor $\mathcal{L}_{n,2}(2^{n-2}, 1^5)$. Let us denote by $\mathcal{L}'$ the kernel of the restriction map $\mathcal{L} \to \mathcal{L}|_{\mathrm{J}(I,C)}$. We can write

$$\mathcal{L}' = \mathcal{L}_{n,d'}(m_1', \ldots, m_{n+3}') := \mathcal{L}_{n,d-2}(m_1-2, \ldots, m_{n-2}-2, m_{n-1}-1, \ldots, m_{n+3}-1),$$

and the inequality $\dim(\mathcal{L}) \geq \dim(\mathcal{L}')$ is satisfied.

Notice that if $m_{n-2} = 2$, i.e. $m_{n-2}' = 0$, then $\mathcal{L}'$ is based on at most $n+2$ points. In this case we have

$$\begin{aligned}
\sum_{i=1}^{n+3} m_i' - nd' &= (M - 2s - 1) - n(d-2) \\
&= (M - m_1 - m_{n-2} - (n-1)d) + (m_{n-2} + m_1 - d - 1) \leq 0.
\end{aligned}$$

The inequality follows by the fact that $\mathcal{L}$ is Cremona reduced and that $m_1 \leq d-1$. One concludes by noticing that $\mathcal{L}'$ falls into case (0).

Otherwise, if $m_{n-2} \geq 3$, since we also have $m_{n+3} \geq 2$, i.e. $m_{n+3}' \geq 1$, then $\mathcal{L}'$ is based on $n+3$ points. We claim that such a $\mathcal{L}'$ satisfies conditions (A_n), (B_n), $(C_{n,t})$. Moreover $k_C' = k_C - 1$, namely the multiplicity of containment of C in the base locus of $\mathcal{L}'$ has decreased by one. If $k_C' = 0$ we conclude by case (0), otherwise we proceed with cases (1) or (2).

We are now left with showing the claim. One can easily check that the first two conditions are satisfies, because of the assumption $m_i \leq d-1$. In order to prove that the third set of conditions, $(C_{n,t})$, is also satisfied for any set $I = I(n-2t)$, $n \geq 2t-1$, notice that

$$\sum_{i\in I} m_i' \leq \sum_{i\in I} m_i - (n - 2t + 1 + f),$$

where f is the cardinality of the index set $I \cap \{1, \ldots, n-2\}$. From this, it follows that

$$tM' + \sum_{i\in I} m_i' - ((t+1)n-t)d' \leq tM + \sum_{i\in I} m_i - ((t+1)n-t)d + (n-t-f-1).$$

Now, choose $2t$ distinct indices $\{i_1,\ldots,i_{2t}\}\subset\{1,\ldots,n+3\}\setminus I$; the right hand side of the above expression equals

$$\sum_{j=1}^{t}\big(M-m_{i_j}-m_{i_{j+t}}-(n-1)d\big)+\alpha,$$

where

$$\alpha:=\left(\sum_{j=1}^{2t}m_{i_i}+\sum_{i\in I}m_i-nd\right)+(n-t-f-1).$$

Here we introduce the integer α for the sake of simplicity as we will treat different cases in what follows. Notice that because of the assumption that $\mathcal{L}$ is Cremona reduced, in order to conclude it is enough to prove that $\alpha\leq 0$.

Assume $d\geq n-t-1$. We have

$$\alpha=\left(\sum_{j=1}^{2t}m_{i_j}+\sum_{i\in I}m_i-(n-1)d\right)+(n-t-f-1-d)\leq 0,$$

where the inequality follows from the fact that $\mathcal{L}$ is Cremona reduced.

If $d\leq n-t-2$, using $m_i\leq d-1$ we obtain

$$\alpha\leq(n+1)(d-1)-nd+(n-t-f-1)=n-2t-4-f\leq 0,$$

where the last inequality is implied by the fact that $f\geq\min\{0,n-2t-4\}$.

Case (2.b). Assume that $\mathcal{L}$ is not Cremona reduced, namely that

$$c:=\sum_{i=1}^{n+1}m_i-(n-1)d\geq 1$$

and write

$$\mathcal{L}':=\mathrm{Cr}(\mathcal{L})=\mathcal{L}_{n,d'}(m_1',\ldots,m_{n+3}').$$

We have $\dim(\mathcal{L})=\dim(\mathcal{L}')$ by (3.4). We claim that $\mathcal{L}'$ satisfies conditions (A_n), (B_n), $(C_{n,t})$. Hence we can reiterate the entire procedure for $\mathcal{L}'$, hence reducing the proof of the effectivity of $\mathcal{L}$ to the proof of the effectivity of its Cremona transform $\mathcal{L}'$.

We now prove the claim. We refer to conditions (A_n), (B_n) and $(C_{n,t})$ for $\mathcal{L}'$ as $(A_n)'$, $(B_n)'$ and $(C_{n,t})'$.

Notice that $m_i'\leq d'$ if and only if $m_i\leq d$, for all $i\leq n+1$. Moreover $m_i'\leq d'$ is equivalent to (B_n), for $i=n+2,n+3$.

One can easily check that (B_n) implies $(B_n)'$.

We now prove that $(C_{n,t})'$ is satisfied for any t and any index set $I=I(n-2t)$, with $n\geq 2t-1$. The expanded expression of condition $(C_{n,t})'$ is

$$tM'+\sum_{i\in I}m_i'-((t+1)n-t)d'\leq 0.$$

Assume that $I\subset\{1,\ldots,n+1\}$. The left-hand side of the above expression equals

$$tM+\sum_{i\in I}m_i-((t+1)n-t)d-c$$

that is negative because $c\geq 1$ and $(C_{n,t})$ is satisfied.

Assume that $|I\setminus\{1,\ldots,n+1\}|=1$, that occurs only if $n\geq 2t$. One can easily verify that $(C_{n,t})'$ is equivalent to $(C_{n,t})$.

Assume that $|I \setminus \{1,\dots,n+1\}| = 2$, that occurs only when $n \geq 2t+1$. The left-hand side of the expanded expression of condition $(C_{n,t})'$ equals

$$tM + \sum_{i\in I} m_i - ((t+1)n-t)d + c = (t+1)M + \sum_{i\in I(n-2t-2)} m_i - ((t+2)n-(t+1))d$$

and this is bounded above by zero by $(C_{n,t+1})$. □

Proof of Theorem 5.1, part (II). Notice that the arithmetic condition $(C_{n,-1})$ corresponds to $d \geq 0$ and $(C_{n,0})$ corresponds to $M - m_i - m_j \leq nd$. When multiplicities m_i are positive, condition $(C_{n,0})$ is redundant and the statement was proved in part *(I)*.

We assume now $m_1 \geq \cdots \geq m_{n+3}$ and $m_{n+3} < 0$. Then conditions $(C_{n,-1})$ and $(C_{n,0})$ imply $(C_{n,t})$ for $t \geq 1$. Moreover the non-emptiness of $\mathcal{L}$ is equivalent to the non-emptiness of $\mathcal{L}_{n,d}(m_1,\dots,m_{n+2})$. This holds because every effective divisor D in $\mathcal{L}$ decomposes as $D = -m_{n+3}E_{n+3} + (D + m_{n+3}E_{n+3})$. The statement follows now from the description of the effective cone of divisors on the blown-up $\mathbb{P}^n$ at $s \leq n+2$ points, that is given by (A_n), (B_n), $(C_{n,-1})$ and $(C_{n,0})$, see Corollary 2.5. □

5.1. **Movable Cone of Divisors.** *Mori dream spaces* were introduced by Hu and Keel ([26], see also [2]), we give now an alternative definition. Let X be a normal $\mathbb{Q}$-factorial variety whose Picard group, $\mathrm{Pic}(X)$, is a lattice. Define the *Cox ring* of X as

$$\mathrm{Cox}(X) := \bigoplus_{D\in \mathrm{Pic}(X)} H^0(X,D)$$

with multiplicative structure defined by a choice of divisors whose classes form a basis for the Picard group $\mathrm{Pic}(X)$. We say that X is a Mori dream space if the Cox ring, $\mathrm{Cox}(X)$, is finitely generated.

We define the *movable cone* of a variety, $\mathrm{Mov}(X)$, to be the cone generated by divisors without divisorial base locus. For a Mori dream space, the movable cone and the effective cone of divisors are polyhedral. Moreover, the movable cone decomposes into disjoint union of *nef chambers* that are the nef cones of all small $\mathbb{Q}$-factorial modifications.

Let X now denote the blow-up of the projective space $\mathbb{P}^n$ at $s \leq n+3$ points in general position. We first recall that X is a Mori dream space [10] (see also [31]). Moreover results of [8, 10] imply that the effective cone $\mathrm{Eff}_{\mathbb{R}}(X)$ is generated as a cone and semigroup by divisors in the *Weyl orbit* $W \cdot E_i$. Here, W represents the *Weyl group* of X and $W \cdot E_i$ the orbit with respect to its action on an exceptional divisor. Recall that every element of W corresponds to a birational map of $\mathbb{P}^n$ lying in the group generated by projective automorphisms and standard Cremona transformations of $\mathbb{P}^n$. Also,

$$\mathrm{Pic}(X) = \langle H, E_1, E_2, \dots, E_s\rangle.$$

Following [30] we introduce a symmetric bilinear on $\mathrm{Pic}(X)$ acting on its generators as:

$$\langle E_i, E_j\rangle = -\delta_{i,j},\ \langle E_i, H\rangle = 0,\ \langle H, H\rangle = n-1. \tag{5.1}$$

Let $\mathrm{Eff}_{\mathbb{R}}(X)^\vee$ denote the dual cone of the cone of effective divisors with real coefficients. Namely, the dual cone $\mathrm{Eff}_{\mathbb{R}}(X)^\vee$ consists of divisors D such that

$\langle D, F\rangle \geq 0$ with all $F \in \mathrm{Eff}_{\mathbb{R}}(X)$. One can define the *degree* of a divisor $D \in \mathrm{Pic}(X)$ as follows (see [10]):

$$\deg(D) := \frac{1}{n-1}\langle D, -K_X\rangle,$$

where K_X denotes the canonical divisor of the blown-up projective space, X. For $s \leq n+3$ the movable cone can be described as the intersection between the cone of effective divisors with real coefficients and its dual

$$\mathrm{Mov}(X) = \mathrm{Eff}_{\mathbb{R}}(X) \cap \mathrm{Eff}_{\mathbb{R}}(X)^{\vee}, \tag{5.2}$$

(see [8, Theorem 4.7]).

The facets of the effective cone $\mathrm{Eff}_{\mathbb{R}}(X)$ for $s \leq n+2$ are given in Corollary 2.5. As an application of this result and using (5.2), one can easily describe the facets of the movable cone $\mathrm{Mov}(X)$ for $s \leq n+2$ (see [8]).

We will extend this description to the case with $s = n+3$ points. The facets of the effective cone $\mathrm{Eff}_{\mathbb{R}}(X)$ of the blown-up $\mathbb{P}^n$ at $n+3$ points in general position are computed in Theorem 5.1. Moreover the generators of the effective cone are described in [10] by the classes of divisors in the set

$$\mathcal{A} = \left\{(t+1)H - (t+1)\sum_{i\in I} E_i - t\sum_{i\notin I} E_i : |I| = n-2t, -1 \leq t \leq l+\epsilon\right\}, \tag{5.3}$$

where $n = 2l+\epsilon$, $\epsilon \in \{0,1\}$. Notice that the divisors in $\mathcal{A}$ are the only divisors on X of degree 1 and are the strict transforms of the one-section linear systems described in (3.3); these are precisely the divisors that will appear in Conjecture 6.4.

We can now describe the movable cone of divisors on the blown-up $\mathbb{P}^n$ at the collection of $n+3$ points. They are the effective divisors on X for which the corresponding linear system has no fixed divisorial component of type $\mathcal{A}$.

Theorem 5.3. *For $n \geq 2$, letting $(d, m_1, \ldots, m_{n+3})$ be the coordinates of the Neron-Severi group, $\mathrm{N}^1(X)$, then the movable cone $\mathrm{Mov}(X)$ is generated by the inequalities (A_n) and (B_n), of Theorem 5.1 and*

$$(D_{n,t}) \qquad k_{I,\sigma_t} \leq 0, \qquad \forall |I| = n-2t,\ -1 \leq t \leq l+\epsilon.$$

Proof. It follows from Theorem 5.1, [8, Theorem 4.7] and [10, Theorem 2.7]. Indeed a divisor in $\mathrm{N}^1(X)$ of the form

$$D = dH - \sum_{i=1}^{n+3} m_i E_i$$

with $d, m_i \geq 0$ lies in $\mathrm{Eff}_{\mathbb{R}}(X)^{\vee}$ if and only if it has non-negative intersection number (5.1) with all elements of the generating set $\mathcal{A}$ described in (5.3). We leave it to the reader to verify that these conditions are equivalent to the set of inequalities $(D_{n,t})$. Notice also that the conditions $(C_{n,t})$ in Theorem 5.1 are redundant. One concludes the proof by using (5.2). $\square$

5.2. **Faces of the movable cone and contractions.** From Mori theory it follows that the faces of the movable cone are in one to one correspondence with classes of divisorial and fibre type contractions from small $\mathbb{Q}$-factorial modifications of X to normal projective varieties.

In particular, contractions given by divisors in the boundary of the effective cone, corresponding to the first three sets of equalities, namely (A_n) and (B_n), are of fibre type contractions (i.e. projections to lower dimensional Mori dream spaces), while contractions associated to the last set of equalities, namely $(D_{n,t})$, corresponding to the boundary of the dual effective cone, are divisorial contractions.

6. A new notion of expected dimension

Secant varieties to the rational normal curve and cones over them are a natural generalization of the linear obstructions. In this section we introduce a new notion of expected dimension for linear systems with $n+3$ points in general position, Definition 6.1, that takes into account their contributions. Furthermore we conjecture that those are the only non-linear obstructions, see Conjecture 6.4.

In Section 6.2 we prove this conjecture for $n \leq 3$ and for some homogeneous linear systems in families.

We adopt the same notation as in the previous sections (3.2) and (4.2). We recall here that the join $\mathrm{J}(I, \sigma_t)$ has dimension $r_{I,\sigma_t} \leq n-1$ whenever $0 \leq t \leq l+\epsilon$, $n = 2l+\epsilon$ and $0 \leq |I| \leq n-2t$.

Definition 6.1. Let $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_{n+3})$ be a linear system. The *(affine) secant linear virtual dimension* of $\mathcal{L}$ is the number

$$\sum_{I,\sigma_t} (-1)^{|I|} \binom{n + k_{I,\sigma_t} - r_{I,\sigma_t} - 1}{n}, \tag{6.1}$$

where the sum ranges over all indexes $I \subset \{1, \ldots, n+3\}$ and t such that $0 \leq t \leq l+\epsilon$, $n = 2l+\epsilon$ and $0 \leq |I| \leq n-2t$.

The *(affine) secant linear expected dimension* of $\mathcal{L}$, denoted by $\sigma\mathrm{ldim}(\mathcal{L})$ is defined as follows: if the linear system $\mathcal{L}$ is contained in a linear system whose secant linear virtual dimension is negative, then we set $\sigma\mathrm{ldim}(\mathcal{L}) = 0$, otherwise we define $\sigma\mathrm{ldim}(\mathcal{L})$ to be the maximum between the secant linear virtual dimension of $\mathcal{L}$ and 0.

Remark 6.2. Using the base locus lemma (Lemma 4), one may generalise formula (6.1) for arbitrary number of points, by taking into account all of the rational normal curves of degree n given by sets of $n+3$ points (and related cycles).

Remark 6.3. One can easily verify that $k_{I,\sigma_t} \leq r_{I,\sigma_t}$, so its corresponding Newton binomial in (6.1) is zero. In particular one can check that this inequality is satisfied for all I and t when $k_C \leq 1$, namely when the rational normal curve C is contained in the base locus of $\mathcal{L}$ at most simply. In all of these cases $\sigma\mathrm{ldim}(\mathcal{L}) = \mathrm{ldim}(\mathcal{L})$.

Conjecture 6.4. *Let $\mathcal{L}$ be a non-empty linear system of $\mathbb{P}^n$ with $n+3$ base points in general position and let C be the rational normal curve through the base points. Then $\mathcal{L}$ is special only if its base locus contains either linear cycles, or cones over the secant varieties σ_t of C. Moreover, we have* $\dim(\mathcal{L}) = \sigma\mathrm{ldim}(\mathcal{L})$.

We illustrate this idea in the following examples.

Example 6.5. The linear system $\mathcal{L} = \mathcal{L}_{6,8}(6^9)$ is linearly special, since $\dim(\mathcal{L}) = 1$ and $\mathrm{ldim}(\mathcal{L}) = -147$. The rational curve C, given by the 9 base points, is contained in the singular locus of the fixed hypersurface $\mathcal{L}$ with multiplicity $k_C = 6$. Moreover, for each of the 9 base points, say p, the cone $\mathrm{J}(p, C)$ as well as σ_2 are contained with

multiplicity 4 in the singular locus of $\mathcal{L}$, by Lemma 4.1. Hence one can compute σldim $= 1$.

Example 6.6. Consider the linear system $\mathcal{L}_{4,10}(9, 7^3, 5^3)$. The rational normal curve is contained 5 times and the cone $J(p_1, C)$ is contained with multiplicity 4 We leave it to the reader to verify that $\sigma\mathrm{ldim}(\mathcal{L}) = 2$ and that $\dim(\mathcal{L}) = 2$, the last equality following a series of Cremona transformations (see Section 3.3).

6.1. **Properties of σldim.** In this section we prove two technical lemma which will be useful in the sequel.

Recall that a linear system $\mathcal{L} = \mathcal{L}_{n,d}(d, m_2, \ldots, m_s)$ has the same dimension of the linear system $\mathcal{L}_{n-1,d}(m_2, \ldots, m_s)$. We will call the second system the *cone reduction* of $\mathcal{L}$ and we denote it by $\mathrm{Cone}(\mathcal{L})$.

Lemma 6.7. *The secant linear expected dimension of a linear system $\mathcal{L} = \mathcal{L}_{n,d}(d, m_2, \ldots, m_{n+3})$ is invariant under cone reduction:*

$$\sigma\mathrm{ldim}(\mathcal{L}) = \sigma\mathrm{ldim}(\mathrm{Cone}(\mathcal{L})).$$

Proof. Let $\mathrm{Cone}(\mathcal{L}) = \mathcal{L}_{n-1,d}(m_2, \ldots, m_{n+3})$ be the cone reduction of $\mathcal{L}$.

We write the formula (6.1) for $\sigma\mathrm{ldim}(\mathcal{L})$ as follows:

$$\sigma\mathrm{ldim}(\mathcal{L}) = \sum_{I,t} B_{\mathcal{L}}(\mathrm{J}(L_I, \sigma_t))$$

denoting by $B_{\mathcal{L}}(\mathrm{J}(L_I, \sigma_t))$ the contribution in the sum given by the cycle $\mathrm{J}(L_I, \sigma_t)$ that is

$$B_{\mathcal{L}}(\mathrm{J}(L_I, \sigma_t)) := (-1)^{|I|}\binom{n + k_{I,\sigma_t} - r_{I,\sigma_t} - 1}{n}.$$

Now, recalling the formula $\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1}$, it is easy to check that, for any $I \subseteq \{2, \ldots, n+3\}$, one has

$$B_{\mathcal{L}}(\mathrm{J}(L_I, \sigma_t)) + B_{\mathcal{L}}(\mathrm{J}(L_{I\cup\{1\}}, \sigma_t)) = B_{\mathrm{Cone}(\mathcal{L})}(\mathrm{J}(L_I, \sigma_t)).$$

□

Lemma 6.8. *The secant linear expected dimension of a linear system $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_{n+3})$ is invariant under Cremona transformations:*

$$\sigma\mathrm{ldim}(\mathcal{L}) = \sigma\mathrm{ldim}(\mathrm{Cr}(\mathcal{L})).$$

Proof. Assume that

$$c = \sum_{i=1}^{n+1} m_i - (n-1)d \geq 1$$

and let

$$\mathrm{Cr}(\mathcal{L}) = \mathcal{L}_{n,d-c}(m_1 - c, \ldots, m_{n+1} - c, m_{n+2}, m_{n+3}).$$

be the system obtained after the Cremona transformation, see Section 3.3.

First of all consider the linear system obtained from $\mathcal{L}$ forgetting the last two points: $\widetilde{\mathcal{L}} = \mathcal{L}_{n,d}(m_1, \ldots, m_{n+1})$ and let $\mathrm{Cr}(\widetilde{\mathcal{L}}) = \mathcal{L}_{n,d-c}(m_1 - c, \ldots, m_{n+1} - c)$ be the corresponding Cremona transform. Since $\widetilde{\mathcal{L}}$ and $\mathrm{Cr}(\widetilde{\mathcal{L}})$ are linearly non-special by Theorem 2.3 and a Cremona transformation preserves the dimension of a linear system (see (3.4)) we have:

$$\mathrm{ldim}(\widetilde{\mathcal{L}}) = \dim(\widetilde{\mathcal{L}}) = \dim(\mathrm{Cr}(\widetilde{\mathcal{L}})) = \mathrm{ldim}(\mathrm{Cr}(\widetilde{\mathcal{L}})).$$

Using the same notation as in the proof of Lemma 6.7, one can split the sum as follows:

$$\begin{aligned}\sigma\mathrm{ldim}(\mathcal{L}) = \mathrm{ldim}(\widetilde{\mathcal{L}}) &+ \sum_{|I\cap\{n+2,n+3\}|=1} B_{\mathcal{L}}(\mathrm{J}(L_I,\sigma_t)) \\ &+ \sum_{|I\cap\{n+2,n+3\}|=2} B_{\mathcal{L}}(\mathrm{J}(L_I,\sigma_t)) \\ &+ \sum_{t\geq 1, I\cap\{n+2,n+3\}=\emptyset} B_{\mathcal{L}}(\mathrm{J}(L_I,\sigma_t)),\end{aligned}$$

and similarly

$$\begin{aligned}\sigma\mathrm{ldim}(\mathrm{Cr}(\mathcal{L})) = \mathrm{ldim}(\mathrm{Cr}(\widetilde{\mathcal{L}})) &+ \sum_{|I\cap\{n+2,n+3\}|=1} B_{\mathrm{Cr}(\mathcal{L})}(\mathrm{J}(L_I,\sigma_t)) \\ &+ \sum_{|I\cap\{n+2,n+3\}|=2} B_{\mathrm{Cr}(\mathcal{L})}(\mathrm{J}(L_I,\sigma_t)) \\ &+ \sum_{t\geq 1, I\cap\{n+2,n+3\}=\emptyset} B_{\mathrm{Cr}(\mathcal{L})}(\mathrm{J}(L_I,\sigma_t)).\end{aligned}$$

Now it is not difficult to check that $\sigma\mathrm{ldim}(\mathrm{Cr}(\mathcal{L})) = \sigma\mathrm{ldim}(\mathcal{L})$. Indeed first of all we note that

$$k_C^c = \sum_{i=1}^{n+3} m_i - c(n+1) - n(d-c) = k_C - c = m_{n+2} + m_{n+3} - d, \tag{6.2}$$

where we denote by k_C (resp. k_C^c) the multiplicity of containment of C in the base locus of $\mathcal{L}$ (resp. of $\mathrm{Cr}(\mathcal{L})$).

Let us also denote by k_{I,σ_t} (resp. k^c_{I,σ_t}) the multiplicity of containment of $\mathrm{J}(L_I,\sigma_t)$ in the base locus of $\mathcal{L}$ (resp. of $\mathrm{Cr}(\mathcal{L})$). We leave it to the reader to check by using (6.2) that the following holds.

If $|I\cap\{n+2,n+3\}| = 1$, then $k^c_{I,\sigma_t} = k_{I,\sigma_t}$, hence $B_{\mathrm{Cr}(\mathcal{L})}(\mathrm{J}(L_I,\sigma_t)) = B_{\mathcal{L}}(\mathrm{J}(L_I,\sigma_t))$.

If $|I\cap\{n+2,n+3\}| = 2$, then $k^c_{I,\sigma_t} = k_{I\setminus\{n+2,n+3\},t+1}$, hence $B_{\mathrm{Cr}(\mathcal{L})}(\mathrm{J}(L_I,\sigma_t)) = B_{\mathcal{L}}(\mathrm{J}(L_{I\setminus\{n+2,n+3\}},\sigma_{t+1}))$.

If $t\geq 1$ and $I\cap\{n+2,n+3\}=\emptyset$, then $k^c_{I,\sigma_t} = k_{I\cup\{n+2,n+3\},t-1}$, hence $B_{\mathrm{Cr}(\mathcal{L})}(\mathrm{J}(L_I,\sigma_t)) = B_{\mathcal{L}}(\mathrm{J}(L_{I\cup\{n+2,n+3\}},\sigma_{t-1}))$. □

6.2. **Cases where Conjecture 6.4 holds.** In this section we provide a list of evidences to Conjecture 6.4.

6.2.1. *Conjecture 6.4 holds for Cremona transforms of only linearly obstructed linear systems.*

Proposition 6.9. *Let $\mathcal{L}$ be linear system with $n+3$ base points for which $k_C\leq 1$. Any linear system $\mathcal{L}'$ that can be Cremona reduced to $\mathcal{L}$ satisfies Conjecture 6.4.*

Proof. We have $\sigma\mathrm{ldim}(\mathcal{L}') = \sigma\mathrm{ldim}(\mathcal{L}) = \mathrm{ldim}(\mathcal{L}) = \dim(\mathcal{L})$. The first equality follows from Lemma 6.8, the second follows from Definition 6.1 (see also Remark 6.3), the last inequality follows from Theorem 2.3. □

6.2.2. *Conjecture 6.4 is true for $n \leq 3$.*

Proposition 6.10. *Conjecture 6.4 holds for $n = 2$.*

Proof. Set $\mathcal{L} = \mathcal{L}_{2,d}(m_1, \ldots, m_5)$. It is a well-known fact that the Segre-Harbourne-Gimigliano-Hirschowitz Conjecture holds for five points. Moreover, from Riemann-Roch Theorem on the blow-up projective plane it follows that

$$\dim(\mathcal{L}) = \binom{d+2}{2} - \sum_{1}^{5} \binom{m_i+1}{2} + \sum_{i,j} \binom{m_i+m_j-d}{2} + \binom{k_C}{2}$$

and one can easily verify that the right-hand side of the above is $\sigma\mathrm{ldim}(\mathcal{L})$. □

Proposition 6.11. *Conjecture 6.4 holds true in $n = 3$.*

Proof. Let $\mathcal{L}$ be a linear system in $\mathbb{P}^3$. If $\mathcal{L}$ is Cremona reduced, then it is linearly non-special by [19, Theorem 5.3], that is $\dim(\mathcal{L}) = \mathrm{ldim}(\mathcal{L})$. On the other hand since $k_C \leq 0$ by Remark 6.3 we have $\sigma\mathrm{ldim}(\mathcal{L}) = \mathrm{ldim}(\mathcal{L})$ and this concludes the proof in this case.

Assume now that $\mathcal{L}$ is not Cremona reduced and denote by $\mathrm{Cr}(\mathcal{L})$ the corresponding Cremona reduced linear system. We have $\sigma\mathrm{ldim}(\mathcal{L}) = \sigma\mathrm{ldim}(\mathcal{L}') = \dim(\mathcal{L}') = \dim(\mathcal{L})$. The first equality follows from Lemma 6.8, the second follows from the previous case and the last one from (3.4). □

6.2.3. *Families of homogeneous linear systems that satisfy Conjecture 6.4.* Consider the following family of linear systems

$$\mathcal{L}(t, a) := \mathcal{L}_{2t, a(t+1)}((at)^{2t+3}),$$

for all $t, a \geq 1$. Notice that in the case $a = 1$, the linear system has one section that is σ_t, see Section 3.2.

Proposition 6.12. *For any $t, a \geq 1$, $\mathcal{L} = \mathcal{L}(t, a)$ has one element, $a\sigma_t$. In particular it satisfies Conjecture 6.4.*

Proof. The hypersurface $a\sigma_t$ belongs to $\mathcal{L}(t, a)$ because it has degree $a(t + 1)$ and multiplicity at along the rational normal curve given by the $2t + 3$ points, see discussion in Section 3.

We prove by induction on t that $a\sigma_t$ is the unique element of $\mathcal{L}(t, a)$ and that $\sigma\mathrm{ldim}(\mathcal{L}) = 1$. If $t = 1$, then the system $\mathcal{L} = \mathcal{L}_{2,2a}(a^5)$ has one section that consists of the multiple conic $a\sigma_1 \subset \mathbb{P}^2$. Furthermore it is easy to compute that $\sigma\mathrm{ldim}(\mathcal{L}) = 1$.

Now assume that $t \geq 2$. First, by means of a Cremona transformation we reduce to $\mathrm{Cr}(\mathcal{L}(a, t)) = \mathcal{L}_{2t,at}((a(t-1))^{2t+1}, (at)^2)$ and $\sigma\mathrm{ldim}(\mathcal{L}(a, t)) = \sigma\mathrm{ldim}(\mathrm{Cr}(\mathcal{L}(a, t)))$, by Lemma 6.8. Second, we observe that $\mathrm{Cone}(\mathrm{Cr}(\mathcal{L}(a, t))) = \mathcal{L}(t - 1, a)$. We conclude by induction and by Lemma 6.7. □

For every $b \geq 1$, let us consider the following linear system

$$\mathcal{L}(b) := \mathcal{L}_{n, b(n+2)}((bn)^{n+3}).$$

Proposition 6.13. *Let $n \geq 2$ and $b \geq 1$. The linear system $\mathcal{L}(b)$ has one element if n is even and empty otherwise. In particular $\mathcal{L}(b)$ satisfies Conjecture 6.4.*

Proof. The proof is by induction on $n \geq 1$. If $n = 1$, one has $\dim(\mathcal{L}_{1,3b}(b^4)) = 0$. If $n = 2$, one has $\dim(\mathcal{L}_{2,4b}((2b)^5)) = 1$.

Now assume $n \geq 3$. Notice that $\mathrm{Cone}(\mathrm{Cr}(\mathcal{L}(b))) = \mathcal{L}_{n-2,bn}((b(n-2))^{n+1})$. We conclude by induction on n, using Lemma 6.7 and Lemma 6.8. □

Consider the family

$$\mathcal{L} = \mathcal{L}_{n,d}(n^{n+3}).$$

Proposition 6.14. *The linear system $\mathcal{L}$ satisfies Conjecture 6.4 for any $n \geq 2$ and $d \geq 1$.*

Proof. If $d \leq n+1$, then the system is empty by Theorem 2.4. If $d = n+2$ we conclude by applying Proposition 6.13 in the case $b = 1$. If $d \geq n+3$, the statement follows from Theorem 2.3 and Remark 6.3. □

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